

Hopf bimodules are modules over a diagonal crossed product algebra

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Abstract

If H is a finite dimensional Hopf algebra, C. Cibils and M. Rosso found an algebra X having the property that Hopf bimodules over H^* coincide with left X -modules. We find two other algebras, Y and Z , having the same property; namely, Y is the *two – sided crossed product* $H^* \# (H \otimes H^{op}) \# H^{*op}$ and Z is the *diagonal crossed product* $(H^* \otimes H^{*op}) \bowtie (H \otimes H^{op})$ (both concepts are due to F. Hausser and F. Nill). We also find explicit isomorphisms between the algebras X, Y, Z .

1 Introduction

Let H be a finite dimensional Hopf algebra. In [1] C. Cibils and M. Rosso introduced an algebra $X = (H^{op} \otimes H) \underline{\otimes} (H^* \otimes H^{*op})$ having the property that Hopf bimodules over H^* coincide with left X -modules. This algebra X was further used in [9] by R. Taillefer who proved that if M and N are (finite dimensional) Hopf bimodules over H^* then the Gerstenhaber-Schack cohomology groups $H_{GS}^*(M, N)$ are isomorphic to $Ext_X^*(M, N)$. The multiplication of X is a “twist” of the one of $(H^{op} \otimes H) \otimes (H^* \otimes H^{*op})$, and it was also proved in [1] that X is isomorphic to the direct tensor product between a Heisenberg double and (the opposite of) a Drinfel’d double.

In this paper we introduce two algebras, Y and Z , with the same property as X (Hopf bimodules over H^* coincide with left modules over Y or Z) but which are more “structured” than X . Namely, Y is the *two – sided crossed product* (in the sense of F. Hausser and F. Nill [3]) $H^* \# (H \otimes H^{op}) \# H^{*op}$, and Z is the *diagonal crossed product* (also in the sense of Hausser and Nill) $(H^* \otimes H^{*op}) \bowtie (H \otimes H^{op})$. We also write down explicit isomorphisms between the algebras X, Y, Z having the

property that if M is a Hopf bimodule over H^* then the actions of X, Y, Z on M correspond via these isomorphisms.

Let us mention that, among these three algebras, our favourite is Z , because the formulae for its multiplication and action on a Hopf bimodule look more elegant than in the other two cases. However, our approach relies on the structure of the algebra Y and the use of “three corners” Hopf modules.

Also, let us note that, as a consequence of the Maschke-type theorem for diagonal crossed products (see [4], Th. 8.2) it follows that if H is semisimple and cosemisimple then Z is semisimple, so we obtain a proof for the semisimplicity of X independent on the isomorphism $X \simeq \mathcal{H}(H^*) \otimes D(H^*)^{op}$.

2 The algebras X, Y, Z

Throughout, k will be a fixed field and all algebras, linear spaces etc. will be over k ; unadorned \otimes means \otimes_k . For coalgebras and Hopf algebras we shall use the framework of [8]. In particular for coalgebras we shall use Σ -notation: $\Delta(h) = \sum h_1 \otimes h_2$, $(id \otimes \Delta)(\Delta(h)) = (\Delta \otimes id)(\Delta(h)) = \sum h_1 \otimes h_2 \otimes h_3$ etc.

In what follows, H will be a finite dimensional Hopf algebra with antipode S . We start by stating some results which are either well-known (see for instance [5]) or easy to prove.

If A is a left H -module algebra, that is A is an algebra and also a left H -module with action denoted by $h \otimes a \mapsto h \cdot a$ and such that $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1_A = \varepsilon(h)1_A$ for all $h \in H$ and $a, b \in A$, the (left) smash product $A \# H$ is the algebra structure on $A \otimes H$ given by

$$(a \# h)(b \# g) = \sum a(h_1 \cdot b) \# h_2 g$$

A left $A \# H$ -module may be identified with a vector space M which is a left H -module and a left A -module (actions denoted by $h \otimes m \mapsto h \cdot m$ and $a \otimes m \mapsto a \cdot m$) related by the following compatibility condition:

$$h \cdot (a \cdot m) = \sum (h_1 \cdot a) \cdot (h_2 \cdot m)$$

for all $h \in H$, $a \in A$, $m \in M$.

If B is a right H -module algebra, that is B is an algebra and also a right H -module with action denoted by $b \otimes h \mapsto b \cdot h$ and such that $(ab) \cdot h = \sum (a \cdot h_1)(b \cdot h_2)$ and $1_B \cdot h = \varepsilon(h)1_B$ for all $h \in H$ and $a, b \in B$, the (right) smash product $H \# B$ is the algebra structure on $H \otimes B$ given by

$$(h \# a)(g \# b) = \sum h g_1 \# (a \cdot g_2) b$$

We are not interested in right $H \# B$ -modules, but also in left $H \# B$ -modules. One can check that a left $H \# B$ -module may be identified with a vector space M which

is a left H -module and a left B -module (actions denoted by $h \otimes m \mapsto h \cdot m$ and $b \otimes m \mapsto b \cdot m$) related by the following compatibility condition:

$$b \cdot (h \cdot m) = \sum h_1 \cdot ((b \cdot h_2) \cdot m)$$

for all $h \in H, b \in B, m \in M$.

If A is a right H -comodule algebra, that is A is an algebra and a right H -comodule with structure map $\rho : A \rightarrow A \otimes H$ which is an algebra map, we can consider the categories ${}_A\mathcal{M}^H$ and \mathcal{M}_A^H of relative Hopf modules (see [5], p. 144). It is well-known (see [2]) that the category ${}_A\mathcal{M}^H$ may be identified with the category of left $A\#H^*$ -modules (A becomes a left H^* -module algebra as usual). Also (see [5]) \mathcal{M}_A^H may be identified with the category of right $A\#H^*$ -modules, but we need to identify it with a category of left modules, and this may be done as follows: since A is a left H^* -module algebra, A^{op} becomes a right H^* -module algebra with action given by $a \cdot p = S(p) \cdot a$ for all $a \in A$ and $p \in H^*$, where we denoted also by S the antipode of H^* . Then one can prove that \mathcal{M}_A^H may be identified with the category of left modules over the right smash product $H^*\#A^{op}$.

We are interested in the categories of Hopf modules ${}_{H^*}^{H^*}\mathcal{M}^{H^*}$ and ${}^{H^*}\mathcal{M}_{H^*}^{H^*}$ (see for instance [7]) and mainly in the category of Hopf bimodules ${}_{H^*}^{H^*}\mathcal{M}_{H^*}^{H^*}$ (introduced for the first time in [6]). We have the obvious identifications:

$$\begin{aligned} {}_{H^*}^{H^*}\mathcal{M}^{H^*} &\simeq {}_{H^*}\mathcal{M}^{H^* \otimes H^{*cop}} \simeq {}_{H^*}\mathcal{M}^{(H \otimes H^{op})^*} \\ {}^{H^*}\mathcal{M}_{H^*}^{H^*} &\simeq \mathcal{M}_{H^*}^{H^* \otimes H^{*cop}} \simeq \mathcal{M}_{H^*}^{(H \otimes H^{op})^*} \end{aligned}$$

where H^* becomes a right $H^* \otimes H^{*cop}$ -comodule algebra via the map $H^* \rightarrow H^* \otimes (H^* \otimes H^{*cop})$, $p \mapsto \sum p_2 \otimes (p_3 \otimes p_1)$.

Let A be a left H -module algebra and B a right H -module algebra (with actions denoted by $h \otimes a \mapsto h \cdot a$ and $b \otimes h \mapsto b \cdot h$). The *two – sided crossed product* $A\#H\#B$, introduced by F. Hausser and F. Nill in [3], is an algebra structure on $A \otimes H \otimes B$, given by

$$(a\#h\#b)(a'\#h'\#b') = \sum a(h_1 \cdot a')\#h_2h'_1\#(b \cdot h'_2)b'$$

(the unit is $1\#1\#1$). Obviously the natural maps from $H, A, B, A\#H, H\#B$ to $A\#H\#B$ are all algebra maps.

Define the category $(A, H, B) - mod$ to be the category whose objects are vector spaces M which are left A -modules, left H -modules and left B -modules, with actions denoted by $a \otimes m \mapsto a \cdot m$, $h \otimes m \mapsto h \cdot m$, $b \otimes m \mapsto b \cdot m$, related by the compatibility conditions:

- (i) $b \cdot (a \cdot m) = a \cdot (b \cdot m)$
- (ii) $b \cdot (h \cdot m) = \sum h_1 \cdot ((b \cdot h_2) \cdot m)$
- (iii) $h \cdot (a \cdot m) = \sum (h_1 \cdot a) \cdot (h_2 \cdot m)$

for all $a \in A, b \in B, h \in H, m \in M$. The morphisms are the maps which are

A -linear, H -linear and B -linear. Let us also note that the conditions (ii) and (iii) above are respectively equivalent to

$$h \cdot (b \cdot m) = \sum (b \cdot S^{-1}(h_2)) \cdot (h_1 \cdot m)$$

$$a \cdot (h \cdot m) = \sum h_2 \cdot ((S^{-1}(h_1) \cdot a) \cdot m)$$

Now we can describe the category of left $A \# H \# B$ -modules.

Proposition 2.1 *There is a natural isomorphism of categories*

$$A \# H \# B - \text{mod} \simeq (A, H, B) - \text{mod}$$

Proof: The identifications are given as follows:

$$a \cdot m = (a \# 1 \# 1) \cdot m$$

$$h \cdot m = (1 \# h \# 1) \cdot m$$

$$b \cdot m = (1 \# 1 \# b) \cdot m$$

and conversely

$$(a \# h \# b) \cdot m = a \cdot (h \cdot (b \cdot m))$$

for all $a \in A, h \in H, b \in B, m \in M$. We shall only prove that the formula $(a \# h \# b) \cdot m = a \cdot (h \cdot (b \cdot m))$ gives indeed a left $A \# H \# B$ -module structure on M provided (i), (ii) and (iii) are satisfied, and leave the rest to the reader. We calculate:

$$(a \# h \# b) \cdot ((a' \# h' \# b') \cdot m) = a \cdot (h \cdot (b \cdot (a' \cdot (h' \cdot (b' \cdot m)))))$$

$$= a \cdot (h \cdot (a' \cdot (b \cdot (h' \cdot (b' \cdot m)))))$$

(using (i))

$$= \sum a \cdot (h \cdot (a' \cdot (h'_1 \cdot ((b \cdot h'_2) \cdot (b' \cdot m)))))$$

(using (ii))

$$= \sum a \cdot (h \cdot (a' \cdot (h'_1 \cdot (((b \cdot h'_2) b') \cdot m))))$$

$$= \sum a \cdot ((h_1 \cdot a') \cdot (h_2 h'_1 \cdot (((b \cdot h'_2) b') \cdot m)))$$

(using (iii))

$$= \sum (a(h_1 \cdot a')) \cdot (h_2 h'_1 \cdot (((b \cdot h'_2) b') \cdot m))$$

$$= (\sum a(h_1 \cdot a') \# h_2 h'_1 \# (b \cdot h'_2) b') \cdot m$$

$$= ((a \# h \# b)(a' \# h' \# b')) \cdot m, \text{ q.e.d.} \quad \blacksquare$$

From the above discussion it follows that a left $A \# H \# B$ -module is a left H -module M which is also a left A -module and a left B -module such that $a \cdot (b \cdot m) =$

$b \cdot (a \cdot m)$ for all $a \in A, b \in B, m \in M$ and such that M is also a left $A \# H$ -module and a left $H \# B$ -module.

Define the algebra $Y = H^* \# (H \otimes H^{op}) \# H^{*op}$, where H^* is a left $H \otimes H^{op}$ -module algebra with action

$$(h \otimes h') \cdot f = h \rightharpoonup f \leftharpoonup h'$$

for all $h, h' \in H$ and $f \in H^*$, where \rightharpoonup and \leftharpoonup are the regular actions of H on H^* , given by $(h \rightharpoonup f)(h') = f(h'h)$, $(f \leftharpoonup h')(h) = f(h'h)$, and H^{*op} is a right $H \otimes H^{op}$ -module algebra with action

$$f \cdot (h \otimes h') = S(h \otimes h') \cdot f = (S(h) \otimes S^{-1}(h')) \cdot f = S(h) \rightharpoonup f \leftharpoonup S^{-1}(h')$$

So, the multiplication in Y is given by:

$$\begin{aligned} & (p \# (h \otimes g) \# q)(p' \# (h' \otimes g') \# q') = \\ & = \sum p(h_1 \rightharpoonup p' \leftharpoonup g_1) \# (h_2 h'_1 \otimes g'_1 g_2) \# q'(S(h'_2) \rightharpoonup q \leftharpoonup S^{-1}(g'_2)) \end{aligned}$$

where the multiplications on the last two positions are made in H and H^* (not in H^{op} and H^{*op}).

Now we come to Hopf bimodules. It is clear that a H^* -Hopf bimodule is a left $H \otimes H^{op}$ -module M (i.e. a H^* -bicomodule) which is also an H^* -bimodule and such that M is an object in the categories ${}^{H^*}_{H^*} \mathcal{M}^{H^*}$ and ${}^{H^*} \mathcal{M}^{H^*}_{H^*}$. Regarding the right H^* -module structure of M as a left H^{*op} -module structure, it is clear that $a \cdot (b \cdot m) = b \cdot (a \cdot m)$ for all $a \in A = H^*$ and $b \in B = H^{*op}$.

Now, from the identifications

$${}^{H^*}_{H^*} \mathcal{M}^{H^*} \simeq {}_{H^*} \mathcal{M}^{(H \otimes H^{op})^*} \simeq {}_{H^* \# (H \otimes H^{op})} \mathcal{M}$$

$${}^{H^*} \mathcal{M}^{H^*}_{H^*} \simeq \mathcal{M}^{(H \otimes H^{op})^*}_{H^*} \simeq_{(H \otimes H^{op}) \# H^{*op}} \mathcal{M}$$

and from all the above, we obtain finally:

Proposition 2.2 *There is a natural isomorphism of categories between ${}^{H^*}_{H^*} \mathcal{M}^{H^*}_{H^*}$ and the category of left $H^* \# (H \otimes H^{op}) \# H^{*op}$ -modules.*

We write down explicitly the Y -module structure of a Hopf bimodule $M \in {}^{H^*}_{H^*} \mathcal{M}^{H^*}_{H^*}$. Denote by $p \otimes m \mapsto p \cdot m$ and $m \otimes q \mapsto m \cdot q$ the H^* -bimodule structure of M and by $M \rightarrow H^* \otimes M \otimes H^*$, $m \mapsto \sum m_{(-1)} \otimes m_{(0)} \otimes m_{(1)}$ the H^* -bicomodule structure of M (the left $H \otimes H^{op}$ -module structure of M is then given by $(h \otimes g) \cdot m = \sum m_{(-1)}(g)m_{(1)}(h)m_{(0)}$). Then the action of Y on M is given by:

$$\begin{aligned} & (p \# (h \otimes g) \# q) \cdot m = p \cdot ((h \otimes g) \cdot (m \cdot q)) \\ & = \sum p \cdot ((m \cdot q)_{(-1)}(g)(m \cdot q)_{(1)}(h)(m \cdot q)_{(0)}) \\ & = \sum p \cdot ((m_{(-1)}q_1)(g)(m_{(1)}q_3)(h)m_{(0)} \cdot q_2) \\ & = \sum m_{(-1)}(g_1)q_1(g_2)m_{(1)}(h_1)q_3(h_2)p \cdot m_{(0)} \cdot q_2 \end{aligned}$$

$$= \sum m_{(-1)}(g_1)m_{(1)}(h_1)p \cdot m_{(0)} \cdot (h_2 \rightharpoonup q \leftharpoonup g_2)$$

for all $p, q \in H^*$, $h, g \in H$ and $m \in M$.

Recall now from [1], [9] the structure of the algebra X of Cibils and Rosso (we write it for H^* , that is in the formulae in [1] and [9] one has to take $A = H^*$). The algebra structure of X is $X = (H^{op} \otimes H) \underline{\otimes} (H^* \otimes H^{*op})$, where the multiplication is defined such that the first two and last two tensorands keep natural multiplication, $(g \otimes h) \underline{\otimes} (p \otimes q) = ((g \otimes h) \underline{\otimes} (1 \otimes 1))((1 \otimes 1) \underline{\otimes} (p \otimes q))$ and

$$\begin{aligned} & ((1 \otimes 1) \underline{\otimes} (p \otimes q))((g \otimes h) \underline{\otimes} (1 \otimes 1)) = \\ &= \sum p_1(S(g_1))p_3(S^{-1}(h_1))q_1(S^{-1}(g_3))q_3(S(h_3))((g_2 \otimes h_2) \underline{\otimes} (p_2 \otimes q_2)) = \\ &= \sum (g_2 \otimes h_2) \underline{\otimes} (S^{-1}(h_1) \rightharpoonup p \leftharpoonup S(g_1) \otimes S(h_3) \rightharpoonup q \leftharpoonup S^{-1}(g_3)) \end{aligned}$$

If M is a Hopf bimodule over H^* , with notation as above, M becomes a left X -module with action given by:

$$((g \otimes h) \underline{\otimes} (p \otimes q)) \cdot m = \sum m_{(-1)}(g_2)m_{(1)}(h_2)(h_1 \rightharpoonup p \leftharpoonup g_1) \cdot m_{(0)} \cdot (h_3 \rightharpoonup q \leftharpoonup g_3)$$

Now, if we look at this formula and the one of the action of Y on M , it is quite clear how to define an algebra isomorphism between X and Y , such that the actions on M correspond.

Proposition 2.3 *The map $\varphi : X \rightarrow Y$, given by*

$$\varphi((g \otimes h) \underline{\otimes} (p \otimes q)) = \sum h_1 \rightharpoonup p \leftharpoonup g_1 \# (h_2 \otimes g_2) \# q$$

is an algebra isomorphism, having the property that

$$((g \otimes h) \underline{\otimes} (p \otimes q)) \cdot m = \varphi((g \otimes h) \underline{\otimes} (p \otimes q)) \cdot m$$

for all $g, h \in H$, $p, q \in H^$ and m in a H^* -Hopf bimodule M . The inverse of φ is given by $\varphi^{-1} : Y \rightarrow X$,*

$$\varphi^{-1}(p \# (h \otimes g) \# q) = \sum (g_2 \otimes h_2) \underline{\otimes} (S^{-1}(h_1) \rightharpoonup p \leftharpoonup S(g_1) \otimes q)$$

Proof: We shall only prove that

$$\begin{aligned} & \varphi(((1 \otimes 1) \underline{\otimes} (p \otimes q))((g \otimes h) \underline{\otimes} (1 \otimes 1))) = \\ &= \varphi((1 \otimes 1) \underline{\otimes} (p \otimes q))\varphi((g \otimes h) \underline{\otimes} (1 \otimes 1)) \end{aligned}$$

and leave the rest of the computations to the reader. We calculate:

$$\begin{aligned} & \varphi(((1 \otimes 1) \underline{\otimes} (p \otimes q))((g \otimes h) \underline{\otimes} (1 \otimes 1))) = \\ &= \varphi(\sum (g_2 \otimes h_2) \underline{\otimes} (S^{-1}(h_1) \rightharpoonup p \leftharpoonup S(g_1) \otimes S(h_3) \rightharpoonup q \leftharpoonup S^{-1}(g_3))) \end{aligned}$$

$$\begin{aligned}
&= \sum (h_2)_1 S^{-1}(h_1) \rightharpoonup p \leftharpoonup S(g_1)(g_2)_1 \# ((h_2)_2 \otimes (g_2)_2) \# S(h_3) \rightharpoonup q \leftharpoonup S^{-1}(g_3) \\
&= \sum p \# (h_1 \otimes g_1) \# S(h_2) \rightharpoonup q \leftharpoonup S^{-1}(g_2) \\
&= (p \# (1 \otimes 1) \# q)(1 \# (h \otimes g) \# 1) \\
&= \varphi((1 \otimes 1) \underline{\otimes} (p \otimes q)) \varphi((g \otimes h) \underline{\otimes} (1 \otimes 1)), \text{ q.e.d.} \quad \blacksquare
\end{aligned}$$

Recall from [3] the definition of the *diagonal crossed product* (which, in a slightly different form, appears also in [10] under the name *right twisted smash product*). If C is an H -bimodule algebra with actions denoted by $h \otimes c \mapsto h \cdot c$ and $c \otimes h \mapsto c \cdot h$, the diagonal crossed product is the following associative algebra structure on $C \otimes H$:

$$(c \otimes h)(c' \otimes h') = \sum c(h_1 \cdot c' \cdot S^{-1}(h_3)) \otimes h_2 h'$$

This structure is denoted by $C \bowtie H$; its unit is $1 \bowtie 1$ and it contains $C \equiv C \bowtie 1$ and $H \equiv 1 \bowtie H$ as subalgebras.

As noted in [10], a linear space M is a left $C \bowtie H$ -module if and only if it is a left H -module and a left C -module with actions $h \otimes m \mapsto h \cdot m$ and $c \otimes m \mapsto c \cdot m$ such that $h \cdot (c \cdot m) = \sum (h_1 \cdot c \cdot S^{-1}(h_3)) \cdot (h_2 \cdot m)$ (the $C \bowtie H$ -module structure of M is then given by $(c \bowtie h) \cdot m = c \cdot (h \cdot m)$).

If A is a left H -module algebra and B is a right H -module algebra, it was proved in [3] that $C = A \otimes B$ is an H -bimodule algebra with actions $h \cdot (a \otimes b) \cdot g = h \cdot a \otimes b \cdot g$ for all $a \in A, b \in B, h, g \in H$, and the map $f : A \# H \# B \rightarrow (A \otimes B) \bowtie H$ given by

$$f(a \# h \# b) = ((a \otimes 1) \bowtie h)((1 \otimes b) \bowtie 1) = \sum (a \otimes b \cdot S^{-1}(h_2)) \bowtie h_1$$

is an algebra isomorphism, with inverse $f^{-1} : (A \otimes B) \bowtie H \rightarrow A \# H \# B$, given by

$$f^{-1}((a \otimes b) \bowtie h) = (1 \# 1 \# b)(a \# h \# 1) = \sum a \# h_1 \# b \cdot h_2$$

From the description of modules over two-sided crossed products and over diagonal crossed products one can see that left modules over $A \# H \# B$ coincide with the ones over $(A \otimes B) \bowtie H$ and that $f(a \# h \# b) \cdot m = (a \# h \# b) \cdot m$ for all $a \in A, b \in B, h \in H$ and $m \in M$, where M is such a module.

We have seen that H^* is a left $H \otimes H^{op}$ -module algebra and H^{*op} is a right $H \otimes H^{op}$ -module algebra, so we can consider the diagonal crossed product $Z = (H^* \otimes H^{*op}) \bowtie (H \otimes H^{op})$, whose multiplication may be written as:

$$\begin{aligned}
&((p \otimes q) \bowtie (h \otimes g))((p' \otimes q') \bowtie (h' \otimes g')) = \\
&= \sum (p(h_1 \rightharpoonup p' \leftharpoonup g_1) \otimes (h_3 \rightharpoonup q' \leftharpoonup g_3)q) \bowtie (h_2 h' \otimes g' g_2)
\end{aligned}$$

(where, as above, the products in the second and fourth positions are in H^* and H , not in H^{*op} and H^{op}).

From the above discussion, the modules over Z are also the same as Hopf bimodules over H^* , and the algebras Y and Z are isomorphic via the maps $\alpha : Y \rightarrow Z$, $\alpha^{-1} : Z \rightarrow Y$,

$$\begin{aligned}\alpha(p\#(h \otimes g)\#q) &= \sum (p \otimes (h_2 \rightharpoonup q \leftharpoonup g_2)) \bowtie (h_1 \otimes g_1) \\ \alpha^{-1}((p \otimes q) \bowtie (h \otimes g)) &= \sum p\#(h_1 \otimes g_1)\#S(h_2) \rightharpoonup q \leftharpoonup S^{-1}(g_2)\end{aligned}$$

Hence, the algebras X and Z are also isomorphic, via the maps $\beta : X \rightarrow Z$, $\beta^{-1} : Z \rightarrow X$, $\beta = \alpha \circ \varphi$, $\beta^{-1} = \varphi^{-1} \circ \alpha^{-1}$, that is

$$\begin{aligned}\beta((g \otimes h) \underline{\otimes} (p \otimes q)) &= \sum (h_1 \rightharpoonup p \leftharpoonup g_1 \otimes h_3 \rightharpoonup q \leftharpoonup g_3) \bowtie (h_2 \otimes g_2) \\ \beta^{-1}((p \otimes q) \bowtie (h \otimes g)) &= \sum (g_2 \otimes h_2) \underline{\otimes} (S^{-1}(h_1) \rightharpoonup p \leftharpoonup S(g_1) \otimes S(h_3) \rightharpoonup q \leftharpoonup S^{-1}(g_3))\end{aligned}$$

and, if M is an H^* -Hopf bimodule, the actions of X and Z on M correspond via these isomorphisms. The action of Z on M is given by

$$((p \otimes q) \bowtie (h \otimes g)) \cdot m = \sum m_{(-1)}(g)m_{(1)}(h)p \cdot m_{(0)} \cdot q$$

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